

A FAMILY OF TWO GENERATOR NON-HOPFIAN GROUPS

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ABSTRACT. We construct 2-generator non-Hopfian groups $G_m, m = 3, 4, 5, \dots$, where each G_m has a specific presentation $G_m = \langle a, b \mid u_{r_{m,0}} = u_{r_{m,1}} = u_{r_{m,2}} = \dots = 1 \rangle$ which satisfies small cancellation conditions $C(4)$ and $T(4)$. Here, $u_{r_{m,i}}$ is the single relator of the upper presentation of the 2-bridge link group of slope $r_{m,i}$, where $r_{m,0} = [m+1, m, m]$ and $r_{m,i} = [m+1, m-1, (i-1)\langle m \rangle, m+1, m]$ in continued fraction expansion for every integer $i \geq 1$.

1. INTRODUCTION

Recall that a group G is called *Hopfian* if every epimorphism $G \rightarrow G$ is an automorphism. The non-Hopfian property of finitely generated groups has a close connection with the non-residual finiteness. In fact, the classical work due to Mal'cev [12] shows that every finitely generated non-Hopfian group is non-residually finite. One of the hardest open problems about hyperbolic groups is whether or not every hyperbolic group is residually finite. An important progress on this problem was given by Sela [19] asserting that every torsion-free hyperbolic group is Hopfian. In 2007, Osin [14] proved that this problem is equivalent to the question on whether or not a group G is residually finite if G is hyperbolic relative to a finite collection of residually finite subgroups. The notion of relatively hyperbolic groups is an important generalization of hyperbolic groups in geometric group theory originally introduced by Gromov [5] (cf. [3], [4], [15]). Motivating examples for this generalization include the fundamental groups of non-compact hyperbolic manifolds of finite volume. In particular, every 2-bridge link complement except for a torus link is a hyperbolic manifold with cusps, so its fundamental group, that is, the 2-bridge link group, is hyperbolic relative to its peripheral subgroups although it is not a hyperbolic group. It is known by Groves [6] that a finitely generated torsion-free group is Hopfian, if it is hyperbolic relative to free abelian subgroups. It is also proved by Reinfeldt and Weidmann [16, 17] that every hyperbolic group, possibly with torsion, is Hopfian. In addition, based on this result, Coulon and Guirardel [2]

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proved that every lacunary hyperbolic group, which is characterized as a direct limit of hyperbolic groups with a certain radii condition, is also Hopfian.

As for small cancellation groups, it is known that if a group has a finite presentation which satisfies small cancellation conditions either $C'(1/6)$ or both $C'(1/4)$ and $T(4)$, then it is hyperbolic (see [20]). Wise [22] also proved that every finite $C'(1/6)$ -small cancellation presentation defines a residually finite group.

Historically, not many have been known examples of finitely generated non-Hopfian groups with specific presentations. The earliest such example was found by Neumann [13] in 1950 as follows: $\langle a, b \mid e_2 = e_3 = \cdots = 1 \rangle$, where $e_i = a^{-1}b^{-1}ab^{-i}ab^{-1}a^{-1}b^i a^{-1}bab^{-i}aba^{-1}b^i$ for every integer $i \geq 2$. Soon after, the first non-Hopfian group with finite presentation was discovered by Higman [7] as follows: $\langle a, s, t \mid a^s = a^2, a^t = a^2 \rangle$. Also a non-Hopfian group with the simplest presentation up to now was produced by Baumslag and Solitar [1] as follows: $\langle a, t \mid (a^2)^t = a^3 \rangle$. Many other non-Hopfian groups with specific finite presentations have been obtained by generalizing Higman's group or Baumslag-Solitar's group (see, for instance, [18], [21]). Another notable non-Hopfian group was obtained by Ivanov and Storozhev [8] in 2005. They constructed a family of finitely generated, but not finitely presented, non-Hopfian relatively free groups with direct limits of hyperbolic groups, although the defining relations of their group presentations are not explicitly described in terms of generators.

Motivated by this background, we construct non-Hopfian groups by using hyperbolic 2-bridge link groups. In more detail, we construct a family of 2-generator non-Hopfian groups each of which has the form $G = \langle a, b \mid u_{r_0} = u_{r_1} = u_{r_2} = \cdots = 1 \rangle$ satisfying small cancellation conditions $C(4)$ and $T(4)$, where u_{r_i} is the single relator of the upper presentation of the link group of the 2-bridge link of slope r_i for every $i = 0, 1, 2, \dots$. Here, the rational numbers r_i may be parametrized by $i \geq 0$, and there is an explicit formula to express u_{r_i} in terms of a and b . To parametrize the rational numbers r_i , we express r_i in continued fraction expansion. Note that every rational number $0 < s \leq 1$ has a unique continued fraction expansion such that

$$s = \frac{1}{m_1 + \frac{1}{m_2 + \cdots + \frac{1}{m_k}}} =: [m_1, m_2, \dots, m_k],$$

where $k \geq 1$, $(m_1, \dots, m_k) \in (\mathbb{Z}_+)^k$ and $m_k \geq 2$ unless $k = 1$.

The main result of the present paper is the following, whose proof is contained in in Section 3.

Theorem 1.1. *Let $r_0 = [4, 3, 3]$, and let $r_i = [4, 2, (i-1)\langle 3 \rangle, 4, 3]$ for every integer $i \geq 1$. Then the group presentation $G = \langle a, b \mid u_{r_0} = u_{r_1} = u_{r_2} = \cdots = 1 \rangle$ satisfies small cancellation conditions $C(4)$ and $T(4)$, and G is non-Hopfian.*

Here, the symbol “ $(i-1)\langle 3 \rangle$ ” represents $i-1$ successive 3’s if $i-1 \geq 1$, whereas “ $0\langle 3 \rangle$ ” means that 3 does not occur in that place, so that $r_1 = [4, 2, 0\langle 3 \rangle, 4, 3] = [4, 2, 4, 3]$.

Remark 1.2. (1) Once we allow the components of a continued fraction expansion to be “ $-$ ”, meaning that the two integers immediately before and after $-$ are added to form one component, r_i ’s in Theorem 1.1 can be parametrized including $i = 0$ as $r_i = [4, i\langle 2, 1, - \rangle, 3, 3]$ for every $i = 0, 1, 2, \dots$.

(2) If we express the rational number r_i in Theorem 1.1 as q_i/p_i , where p_i and q_i are relatively prime positive integers, then $|u_{r_i}| = 2p_i$ (see Section 2.1). A simple computation shows that the inequality $3 < p_{i+1}/p_i < 4$ holds for every $i = 0, 1, 2, \dots$, so that the length $|u_{r_i}|$ of the word u_{r_i} satisfies the inequality $c \cdot 3^i < |u_{r_i}| < c \cdot 4^i$ for every integer $i \geq 1$, where $c = |u_{r_0}|$.

By looking at the proof of Theorem 1.1 in Section 3, it is not hard to see that a similar result holds not only for $r_0 = [4, 3, 3]$ but also for $r_0 = [m+1, m, m]$ with m being any integer greater than 3. Thus we only state its general form without a detailed proof.

Theorem 1.3. *Suppose that m is an integer with $m \geq 3$. Let $r_0 = [m+1, m, m]$, and let $r_i = [m+1, m-1, (i-1)\langle m \rangle, m+1, m]$ for every integer $i \geq 1$. Then the group presentation $G = \langle a, b \mid u_{r_0} = u_{r_1} = u_{r_2} = \dots = 1 \rangle$ satisfies small cancellation conditions $C(4)$ and $T(4)$, and G is non-Hopfian.*

The present paper is organized as follows. In Section 2, we recall the upper presentation of a 2-bridge link group, and basic facts established in [9] concerning the upper presentations. We also recall key facts from [9] obtained by applying small cancellation theory to the upper presentations. Section 3 is devoted to the proof of the main result (Theorem 1.1).

2. PRELIMINARIES

2.1. Upper presentations of 2-bridge link groups

We recall some notation in [9]. The *Conway sphere* \mathbf{S} is the 4-times punctured sphere which is obtained as the quotient of $\mathbb{R}^2 - \mathbb{Z}^2$ by the group generated by the π -rotations around the points in \mathbb{Z}^2 . For each $s \in \hat{\mathbb{Q}} := \mathbb{Q} \cup \{\infty\}$, let α_s be the simple loop in \mathbf{S} obtained as the projection of a line in $\mathbb{R}^2 - \mathbb{Z}^2$ of slope s . We call s the *slope* of the simple loop α_s .

For each $r \in \hat{\mathbb{Q}}$, the *2-bridge link* $K(r)$ of slope r is the sum of the rational tangle $(B^3, t(\infty))$ of slope ∞ and the rational tangle $(B^3, t(r))$ of slope r . Recall that $\partial(B^3 - t(\infty))$ and $\partial(B^3 - t(r))$ are identified with \mathbf{S} so that α_∞ and α_r bound disks in $B^3 - t(\infty)$ and $B^3 - t(r)$, respectively. By van-Kampen’s theorem, the link group $G(K(r)) = \pi_1(S^3 - K(r))$ is obtained as follows:

$$G(K(r)) = \pi_1(S^3 - K(r)) \cong \pi_1(\mathbf{S}) / \langle \langle \alpha_\infty, \alpha_r \rangle \rangle \cong \pi_1(B^3 - t(\infty)) / \langle \langle \alpha_r \rangle \rangle.$$

Let $\{a, b\}$ be the standard meridian generator pair of $\pi_1(B^3 - t(\infty), x_0)$ as described in [9, Section 3]. Then $\pi_1(B^3 - t(\infty))$ is identified with the free group $F(a, b)$ with basis $\{a, b\}$. For the rational number $r = q/p$, where p and q are relatively prime positive integers, let u_r be the word in $\{a, b\}$ obtained as follows. Set $\epsilon_i = (-1)^{\lfloor iq/p \rfloor}$, where $\lfloor x \rfloor$ is the greatest integer not exceeding x .

(1) If p is odd, then

$$u_{q/p} = a \hat{u}_{q/p} b^{(-1)^q} \hat{u}_{q/p}^{-1},$$

$$\text{where } \hat{u}_{q/p} = b^{\epsilon_1} a^{\epsilon_2} \dots b^{\epsilon_{p-2}} a^{\epsilon_{p-1}}.$$

(2) If p is even, then

$$u_{q/p} = a \hat{u}_{q/p} a^{-1} \hat{u}_{q/p}^{-1},$$

$$\text{where } \hat{u}_{q/p} = b^{\epsilon_1} a^{\epsilon_2} \dots a^{\epsilon_{p-2}} b^{\epsilon_{p-1}}.$$

Then $u_r \in F(a, b) \cong \pi_1(B^3 - t(\infty))$ is represented by the simple loop α_r , and we obtain the following two-generator and one-relator presentation of a 2-bridge link groups:

$$G(K(r)) \cong \pi_1(B^3 - t(\infty)) / \langle \langle \alpha_r \rangle \rangle \cong \langle a, b \mid u_r \rangle.$$

This presentation is called the *upper presentation* of a 2-bridge link group.

2.2. Basic facts concerning the upper presentations

Throughout this paper, a *cyclic word* is defined to be the set of all cyclic permutations of a cyclically reduced word. By (v) we denote the cyclic word associated with a cyclically reduced word v . Also the symbol “ \equiv ” denotes the *letter-by-letter equality* between two words or between two cyclic words. Now we recall definitions and basic facts from [9] which are needed in the proof of Theorem 1.1 in Section 3.

Definition 2.1. (1) Let v be a reduced word in $\{a, b\}$. Decompose v into

$$v \equiv v_1 v_2 \dots v_t,$$

where, for each $i = 1, \dots, t-1$, v_i is a positive (resp., negative) subword (that is, all letters in v_i have positive (resp., negative) exponents), and v_{i+1} is a negative (resp., positive) subword. Then the sequence of positive integers $S(v) := (|v_1|, |v_2|, \dots, |v_t|)$ is called the *S-sequence of v*.

(2) Let v be a cyclically reduced word in $\{a, b\}$. Decompose the cyclic word (v) into

$$(v) \equiv (v_1 v_2 \dots v_t),$$

where v_i is a positive (resp., negative) subword, and v_{i+1} is a negative (resp., positive) subword (taking subindices modulo t). Then the *cyclic sequence* of positive integers $CS(v) := (|v_1|, |v_2|, \dots, |v_t|)$ is called the *CS-sequence of (v)*. Here the double parentheses denote that the sequence is considered modulo cyclic permutations.

Definition 2.2. For a rational number r with $0 < r \leq 1$, let u_r be the word defined in the beginning of this section. Then the symbol $CS(r)$ denotes the CS -sequence $CS(u_r)$ of (u_r) , which is called the CS -sequence of slope r .

A reduced word w in $\{a, b\}$ is said to be *alternating* if $a^{\pm 1}$ and $b^{\pm 1}$ appear in w alternately, to be precise, neither $a^{\pm 2}$ nor $b^{\pm 2}$ appears in w . Also a cyclically reduced word w in $\{a, b\}$ is said to be *cyclically alternating*, i.e., all the cyclic permutations of w are alternating. In particular, u_r is a cyclically alternating word in $\{a, b\}$. Note that every alternating word w in $\{a, b\}$ is determined by the sequence $S(w)$ and the initial letter (with exponent) of w . Note also that if w is a cyclically alternating word in $\{a, b\}$ such that $CS(w) = CS(r)$, then either $(w) \equiv (u_r)$ or $(w) \equiv (u_r^{-1})$ as cyclic words.

In the remainder of this section, we suppose that r is a rational number with $0 < r \leq 1$, and write r as a continued fraction expansion $r = [m_1, m_2, \dots, m_k]$, where $k \geq 1$, $(m_1, \dots, m_k) \in (\mathbb{Z}_+)^k$ and $m_k \geq 2$ unless $k = 1$. Note from [9] that if $k \geq 2$, then some properties of $CS(r)$ differ according to $m_2 = 1$ or $m_2 \geq 2$. For brevity, we write m for m_1 .

Lemma 2.3 ([9, Proposition 4.3]). *For the rational number $r = [m_1, m_2, \dots, m_k]$ satisfying that $m_2 \geq 2$ if $k \geq 2$, the following hold.*

- (1) *Suppose $k = 1$, i.e., $r = 1/m$. Then $CS(r) = ((m, m))$.*
- (2) *Suppose $k \geq 2$. Then each term of $CS(r)$ is either m or $m + 1$. Moreover, no two consecutive terms of $CS(r)$ can be $(m + 1, m + 1)$, so there is a cyclic sequence of positive integers $((t_1, t_2, \dots, t_s))$ such that*

$$CS(r) = ((m + 1, t_1 \langle m \rangle, m + 1, t_2 \langle m \rangle, \dots, m + 1, t_s \langle m \rangle)).$$

Here, the symbol " $t_i \langle m \rangle$ " represents t_i successive m 's.

Definition 2.4. If $k \geq 2$, the symbol $CT(r)$ denotes the cyclic sequence $((t_1, t_2, \dots, t_s))$ in Lemma 2.3, which is called the CT -sequence of slope r .

Lemma 2.5 ([9, Proposition 4.4 and Corollary 4.6]). *For the rational number $r = [m_1, m_2, \dots, m_k]$ with $k \geq 2$ and $m_2 \geq 2$, let r' be the rational number defined as*

$$r' = [m_2 - 1, m_3, \dots, m_k].$$

Then we have $CT(r) = CS(r')$.

Lemma 2.6 ([9, Proposition 4.5]). *For the rational number $r = [m_1, m_2, \dots, m_k]$, the cyclic sequence $CS(r)$ has a decomposition $((S_1, S_2, S_1, S_2))$ which satisfies the following.*

- (1) *Each S_i is symmetric, i.e., the sequence obtained from S_i by reversing the order is equal to S_i . (Here, S_1 is empty if $k = 1$.)*
- (2) *Each S_i occurs only twice in the cyclic sequence $CS(r)$.*
- (3) *The subsequence S_1 begins and ends with $m + 1$.*

(4) The subsequence S_2 begins and ends with m .

Lemma 2.7 ([9, Proof of Proposition 4.5]). *For the rational number $r = [m_1, m_2, \dots, m_k]$ with $k \geq 2$ and $m_2 \geq 2$, let r' be the rational number defined as in Lemma 2.5. Also let $CS(r') = ((T_1, T_2, T_1, T_2))$ and $CS(r) = ((S_1, S_2, S_1, S_2))$ be the decompositions described in Lemma 2.6. Then the following hold.*

- (1) *If $k = 2$, then $T_1 = \emptyset$, $T_2 = (m_2 - 1)$, and $S_1 = (m + 1)$, $S_2 = ((m_2 - 1)\langle m \rangle)$.*
- (2) *If $k \geq 3$, then $T_1 = (t_1, \dots, t_{s_1})$, $T_2 = (t_{s_1+1}, \dots, t_{s_2})$, and*

$$S_1 = (m + 1, t_{s_1+1}\langle m \rangle, m + 1, \dots, m + 1, t_{s_2}\langle m \rangle, m + 1),$$

$$S_2 = (t_1\langle m \rangle, m + 1, t_2\langle m \rangle, \dots, t_{s_1-1}\langle m \rangle, m + 1, t_{s_1}\langle m \rangle).$$

The following lemma is useful in the proof of Lemma 3.6.

Lemma 2.8. *For two distinct rational numbers $r = [m_1, m_2, \dots, m_k]$ and $s = [m_1, l_2, \dots, l_t]$, assume that*

- (i) *m is a positive integer;*
- (ii) *m_i and l_j are integers greater than 1 for every $i \geq 2$ and $j \geq 2$;*
- (iii) *$k, t \geq 3$ and $k \neq t$; and*
- (iv) *if $k < t$, then $m_2 \geq l_2$, while if $k > t$, then $m_2 \leq l_2$.*

Let r' and s' be the rational numbers defined as in Lemma 2.5. Also let $CS(r) = ((S_1, S_2, S_1, S_2))$ and $CS(r') = ((T_1, T_2, T_1, T_2))$ be the decompositions described in Lemma 2.6. Suppose that $CS(s)$ contains S_1 or S_2 as a subsequence. Then $CS(s')$ contains T_1 or T_2 as a subsequence.

In the above lemma (and throughout this paper), we mean by a *subsequence* a subsequence without leap. Namely a sequence (a_1, a_2, \dots, a_l) is called a *subsequence* of a cyclic sequence, if there is a sequence (b_1, b_2, \dots, b_n) representing the cyclic sequence such that $l \leq n$ and $a_i = b_i$ for $1 \leq i \leq l$.

Proof. First suppose that $CS(s)$ contains S_1 as a subsequence. By Lemma 2.7(2), $CS(s)$ contains $(m + 1, t_{s_1+1}\langle m \rangle, m + 1, \dots, m + 1, t_{s_2}\langle m \rangle, m + 1)$ as a subsequence, where $T_2 = (t_{s_1+1}, \dots, t_{s_2})$. Then clearly $CS(s') = CT(s)$ contains $(t_{s_1+1}, \dots, t_{s_2})$, that is, T_2 , as a subsequence. So we are done.

Next suppose that $CS(s)$ contains S_2 as a subsequence. Again by Lemma 2.7(2), $CS(s)$ contains $(t_1\langle m \rangle, m + 1, t_2\langle m \rangle, \dots, t_{s_1-1}\langle m \rangle, m + 1, t_{s_1}\langle m \rangle)$ as a subsequence, where $T_1 = (t_1, \dots, t_{s_1})$. Then $CS(s') = CT(s)$ contains $(d_1 + t_1, t_2, \dots, t_{s_1-1}, t_{s_1} + d_2)$ as a subsequence, where $d_1, d_2 \geq 0$. In the reminder of the proof, we show that $d_1 = d_2 = 0$, so that $CS(s') = CT(s)$ contains $T_1 = (t_1, t_2, \dots, t_{s_1})$ as a subsequence. To this end, note that since $r' = [m_2 - 1, m_3, \dots, m_k]$, $t_1 = t_{s_1} = m_2$ by Lemma 2.6(3). Also since $s' = [l_2 - 1, l_3, \dots, l_t]$, $CS(s') = CT(s)$ consists of $l_2 - 1$ and l_2 by Lemma 2.3(2). Hence each of $d_1 + t_1 = d_1 + m_2$ and $t_{s_1} + d_2 = m_2 + d_2$ is either $l_2 - 1$ or l_2 . Suppose first that $k < t$. Then $m_2 \geq l_2$ by the assumption (iv), and thus the only possibility is $m_2 = l_2$. Thus we have $d_1 = d_2 = 0$. Suppose

next that $k > t$. Then $m_2 \leq l_2$ again by the assumption (iv). Note that $k \geq 4$, because $t \geq 3$ by the assumption (iii). Thus we can see, by using Lemma 2.7 and the assumption $m_i \geq 2$ for every $i \geq 2$, that S_2 contains $(m+1, (m_2-1)\langle m \rangle, m+1)$ as a subsequence. This implies that $CS(s') = CT(s)$ contains a term $m_2 - 1$. Since $m_2 - 1 \leq l_2 - 1$, the only possibility $m_2 = l_2$. Thus we again have $d_1 = d_2 = 0$, completing the proof of Lemma 2.8. \square

2.3. Small cancellation theory applied to the upper presentations

A subset R of the free group $F(a, b)$ is called *symmetrized*, if all elements of R are cyclically reduced and, for each $w \in R$, all cyclic permutations of w and w^{-1} also belong to R .

Definition 2.9. Suppose that R is a symmetrized subset of $F(a, b)$. A nonempty word v is called a *piece* (with respect to R) if there exist distinct $w_1, w_2 \in R$ such that $w_1 \equiv vc_1$ and $w_2 \equiv vc_2$. The small cancellation conditions $C(p)$ and $T(q)$, where p and q are integers such that $p \geq 2$ and $q \geq 3$, are defined as follows (see [11]).

- (1) Condition $C(p)$: If $w \in R$ is a product of n pieces, then $n \geq p$.
- (2) Condition $T(q)$: For $w_1, \dots, w_n \in R$ with no successive elements w_i, w_{i+1} an inverse pair ($i \bmod n$), if $n < q$, then at least one of the products $w_1 w_2, \dots, w_{n-1} w_n, w_n w_1$ is freely reduced without cancellation.

The following proposition enables us to apply small cancellation theory to the upper presentation $\langle a, b | u_r \rangle$ of $G(K(r))$.

Proposition 2.10 ([9, Theorem 5.1]). *Let r be a rational number such that $0 < r < 1$, and let R be the symmetrized subset of $F(a, b)$ generated by the single relator u_r of the group presentation $G(K(r)) = \langle a, b | u_r \rangle$. Then R satisfies $C(4)$ and $T(4)$.*

This proposition follows from the following characterization of pieces, which in turn is proved by using Lemma 2.6.

Lemma 2.11 ([10, Corollary 3.25]). *Let r and R be as in Proposition 2.10. Then a subword w of the cyclic word $(u_r^{\pm 1})$ is a piece with respect to R if and only if $S(w)$ contains neither S_1 nor (ℓ_1, S_2, ℓ_2) with $\ell_1, \ell_2 \in \mathbb{Z}_+$ as a subsequence.*

3. PROOF OF THEOREM 1.1

In this section, for brevity of notation, we sometimes write \bar{x} for x^{-1} for a letter or a word x . For a quotient group H of the free group $F(a, b)$ and two elements w_1 and w_2 of $F(a, b)$, the symbol $w_1 =_H w_2$ means the equality in the group H .

For $r_0 = [4, 3, 3]$, we have by using Lemma 2.7

$$CS(u_{r_0}) = CS(r_0) = ((5, 4, 4, 4, 5, 4, 4, 5, 4, 4, 5, 4, 4, 5, 4, 4, 5, 4, 4)).$$

Let $G_0 = \langle a, b \mid u_{r_0} = 1 \rangle$. Also let $X \equiv a \cdots \bar{a}$ be the alternating word in $\{a, b\}$ such that $S(X) = (4, 4, 4, 5, 4, 4)$, and let $f : F(a, b) \rightarrow F(a, b)$ be the homomorphism defined by $f(a) = \bar{X}$ and $f(b) = \bar{b}$.

Lemma 3.1. *Under the foregoing notation, let $\tilde{f} : F(a, b) \rightarrow G_0$ be the composition of f and the canonical surjection $F(a, b) \rightarrow G_0$. Then \tilde{f} is onto.*

Proof. Since $\tilde{f}(b) = \bar{b}$, it suffices to show that $a \in G_0$ is contained in the image of \tilde{f} . Let $w \equiv a \cdots \bar{a}$ be the alternating word in $\{a, b\}$ such that $S(w) = (3, 3, 3, 4, 3, 3)$. Then

$$f(w) = \bar{X}\bar{b}\bar{X}bXb\bar{X}\bar{b}\bar{X}bXbX\bar{b}\bar{X}\bar{b}XbX.$$

Here, since $X \equiv a \cdots \bar{a}$ and $\bar{X} \equiv a \cdots \bar{a}$ are alternating words in $\{a, b\}$, we see that $f(w) \equiv a \cdots \bar{a}$ is also an alternating word in $\{a, b\}$ with

$$S(f(w)) = (S(\bar{X}\bar{b}), S(\bar{X}), S(bX), S(b\bar{X}\bar{b}), S(\bar{X}), S(bX), S(bX\bar{b}), S(\bar{X}\bar{b}), S(X), S(bX)).$$

Since $S(X) = (4, 4, 4, 5, 4, 4)$ and $S(\bar{X}) = (4, 4, 5, 4, 4, 4)$, we have $S(\bar{X}\bar{b}) = (4, 4, 5, 4, 4, 5)$, $S(bX) = (5, 4, 4, 5, 4, 4)$, $S(b\bar{X}\bar{b}) = (5, 4, 5, 4, 4, 5)$ and $S(bX\bar{b}) = (5, 4, 4, 5, 4, 5)$, so that

$$\begin{aligned} S(f(w)) = & (4, 4, 5, 4, 4, 5, 4, 4, 5, 4, 4, 5, 4, 4, 5, 4, 4, 5, 4, 4, 5, 4, 4, 4, \\ & 5, 4, 4, 5, 4, 4, 5, 4, 4, 5, 4, 4, 5, 4, 4, 5, 4, 4, 5, 4, 4, 5, 4, 4). \end{aligned}$$

Letting $v_1 \equiv a \cdots \bar{b}$, $v_2 \equiv a \cdots \bar{b}$, and $v_3 \equiv b \cdots \bar{a}$ be the cyclically alternating words in $\{a, b\}$ such that

$$\begin{aligned} S(v_1) &= (4, 4, 5, 4, 4, 5, 4, 4, 5, 4, 4, 5, 4, 4, 5, 4, 4, 5, 4, 4, 5, 4, 4), \\ S(v_2) &= (5, 4, 4, 5, 4, 4, 5, 4, 4, 5, 4, 4, 5, 4, 4, 5, 4, 4, 5, 4, 4, 4, 4), \\ S(v_3) &= (4, 5, 4, 4, 5, 4, 4, 5, 4, 4, 5, 4, 4, 5, 4, 4, 5, 4, 4, 5, 4, 4, 4), \end{aligned}$$

we see that $f(w) \equiv v_1 v_2 \bar{a} v_3$. Moreover, for each $i = 1, 2, 3$, since $CS(v_i) = CS(r_0)$, $(v_i) \equiv (u_{r_0}^{\pm 1})$ as cyclic words by [9, Lemma 3.2], which implies that $v_i =_{G_0} 1$. Hence $f(w) =_{G_0} \bar{a}$, and thus $a \in G_0$ is contained in the image of \tilde{f} , as required. \square

At this point, we set up the following notation which will be used at the end of the proofs of Lemmas 3.3 and 3.4.

Notation 3.2. (1) Suppose that v is an alternating word in $\{a, b\}$ such that there is a sequence (t_1, t_2, \dots, t_s) of positive integers satisfying

$$S(v) = (\epsilon_1 \langle 5 \rangle, t_1 \langle 4 \rangle, 5, t_2 \langle 4 \rangle, \dots, 5, t_s \langle 4 \rangle, \epsilon_2 \langle 5 \rangle),$$

where ϵ_i is 0 or 1 for $i = 1, 2$. Then the symbol $T(v)$ denotes the sequence (t_1, t_2, \dots, t_s) .

(2) Suppose that v is a cyclically alternating word in $\{a, b\}$ such that there is a cyclic sequence $((t_1, t_2, \dots, t_s))$ of positive integers satisfying

$$CS(v) = ((5, t_1 \langle 4 \rangle, 5, t_2 \langle 4 \rangle, \dots, 5, t_s \langle 4 \rangle)).$$

Then the symbol $CT(v)$ denotes the cyclic sequence $((t_1, t_2, \dots, t_s))$. In particular, by Lemma 2.7, if $v \equiv u_r$ for some $r = [4, m_2, \dots, m_k]$ with $m_2 \geq 2$, then $CT(u_r) = CT(r) = CS(r')$, where $r' = [m_2 - 1, \dots, m_k]$.

(3) Suppose that v is an alternating word in $\{a, b\}$ such that there is a sequence (h_1, h_2, \dots, h_p) of positive integers satisfying

$$T(v) = (\epsilon_1 \langle 2 \rangle, h_1 \langle 1 \rangle, 2, h_2 \langle 1 \rangle, \dots, 2, h_p \langle 1 \rangle, \epsilon_2 \langle 2 \rangle),$$

where $T(v)$ is defined as in (1) and ϵ_i is 0 or 1 for $i = 1, 2$. Then the symbol $V(v)$ denotes the sequence (h_1, h_2, \dots, h_p) .

(4) Suppose that v is a cyclically alternating word in $\{a, b\}$ such that there is a cyclic sequence $((h_1, h_2, \dots, h_p))$ of positive integers satisfying

$$CT(v) = ((2, h_1 \langle 1 \rangle, 2, h_2 \langle 1 \rangle, \dots, 2, h_p \langle 1 \rangle)),$$

where $CT(v)$ is defined as in (2). Then the symbol $CV(v)$ denotes the cyclic sequence $((h_1, h_2, \dots, h_p))$. In particular, by Lemma 2.7, if $v \equiv u_r$ for some $r = [4, 2, m_3, \dots, m_k]$ with $m_3 \geq 2$, then $CV(u_r) = CT(r') = CS(r'')$, where $r' = [1, m_3, \dots, m_k]$ and $r'' = [m_3 - 1, \dots, m_k]$.

Lemma 3.3. *Under the foregoing notation, $(f(u_{r_0})) =_{G_0} (u_{r_1}^{\pm 1})$.*

Proof. Recall that

$$CS(u_{r_0}) = CS(r_0) = ((5, 4, 4, 4, 5, 4, 4, 5, 4, 4, 5, 4, 4, 4, 5, 4, 4, 5, 4, 4)).$$

Clearly the cyclic word (u_{r_0}) has six positive or negative subwords of length 5. Cutting in the middle of such subwords, we may write the cyclic word (u_{r_0}) as a product $(v_1 \cdots v_6)$, where

$$\begin{aligned} v_1 &\equiv abab\bar{a}\bar{b}abab\bar{a}\bar{b}ab, \\ v_2 &\equiv abab\bar{a}\bar{b}abab\bar{a}\bar{b}a, \\ v_3 &\equiv \bar{a}\bar{b}abab\bar{a}\bar{b}ab, \\ v_4 &\equiv abab\bar{a}\bar{b}abab\bar{a}\bar{b}ab, \\ v_5 &\equiv v_2^{-1} \equiv abab\bar{a}\bar{b}abab\bar{a}\bar{b}a, \\ v_6 &\equiv v_3^{-1} \equiv \bar{a}\bar{b}abab\bar{a}\bar{b}ab. \end{aligned}$$

Put $w_n \equiv f(v_n)$ for every $n = 1, \dots, 6$, namely

$$\begin{aligned} w_1 &\equiv \bar{X}\bar{b}\bar{X}bXbX\bar{b}\bar{X}\bar{b}\bar{X}bXbX\bar{b}, \\ w_2 &\equiv \bar{X}\bar{b}\bar{X}\bar{b}XbXb\bar{X}\bar{b}\bar{X}\bar{b}XbX, \\ w_3 &\equiv bX\bar{b}\bar{X}\bar{b}\bar{X}bXbX\bar{b}, \\ w_4 &\equiv \bar{X}\bar{b}\bar{X}\bar{b}XbXb\bar{X}\bar{b}\bar{X}\bar{b}XbXb\bar{X}\bar{b}, \\ w_5 &\equiv w_2^{-1} \quad \text{and} \quad w_6 \equiv w_3^{-1}. \end{aligned}$$

It then follows that

$$(f(u_{r_0})) = (f(v_1 \cdots v_6)) = (w_1 \cdots w_6).$$

Claim 1. $\bar{X}\bar{b}\bar{X}bXbX\bar{b} =_{G_0} z_1$, where $z_1 \equiv a \cdots \bar{b}$ is an alternating word in $\{a, b\}$ with $S(z_1) = (4, 5, 4, 5)$.

Proof of Claim 1. Recall that $X \equiv a \cdots \bar{a}$ and $\bar{X} \equiv a \cdots \bar{a}$ are alternating words in $\{a, b\}$ such that $S(X) = (4, 4, 4, 5, 4, 4)$ and $S(\bar{X}) = (4, 4, 5, 4, 4, 4)$. It is not hard to see that

$$\begin{aligned} S(\bar{X}\bar{b}\bar{X}bXbX\bar{b}) &= (S(\bar{X}\bar{b}), S(\bar{X}), S(bX), S(bX\bar{b})) \\ &= ((4, 4, 5, 4, 4, 5), (4, 4, 5, 4, 4, 4), (5, 4, 4, 5, 4, 4), (5, 4, 4, 5, 4, 5)) \\ &= (4, 4, 5, 4, 4, 5, 4, 4, 5, 4, 4, 5, 4, 4, 5, 4, 4, 5, 4, 4, 5, 4, 5). \end{aligned}$$

Letting $y_1 \equiv a \cdots \bar{b}$ and $z_1 \equiv a \cdots \bar{b}$ be alternating words in $\{a, b\}$ such that $S(y_1) = (4, 4, 5, 4, 4, 5, 4, 4, 5, 4, 4, 5, 4, 4, 5, 4, 4, 5, 4)$ and $S(z_1) = (4, 5, 4, 5)$, clearly $\bar{X}\bar{b}\bar{X}bXbX\bar{b} \equiv y_1 z_1$. Here, since $CS(y_1) = CS(r_0)$ and so $y_1 =_{G_0} 1$, we finally have $\bar{X}\bar{b}\bar{X}bXbX\bar{b} \equiv y_1 z_1 =_{G_0} z_1$, as required. \square

Claim 2. $\bar{X}\bar{b}\bar{X}\bar{b}XbX =_{G_0} z_2$, where $z_2 \equiv a \cdots \bar{a}$ is the alternating word in $\{a, b\}$ with $S(z_2) = (4, 4, 5, 4)$.

Proof of Claim 2. As in the proof of Claim 1, we have

$$\begin{aligned} S(\bar{X}\bar{b}\bar{X}\bar{b}XbX) &= (S(\bar{X}\bar{b}), S(\bar{X}\bar{b}), S(X), S(bX)) \\ &= ((4, 4, 5, 4, 4, 5), (4, 4, 5, 4, 4, 5), (4, 4, 4, 5, 4, 4), (5, 4, 4, 5, 4, 4)) \\ &= (4, 4, 5, 4, 4, 5, 4, 4, 5, 4, 4, 5, 4, 4, 5, 4, 4, 5, 4, 4, 5, 4, 4). \end{aligned}$$

Letting $z_2 \equiv a \cdots \bar{a}$ and $y_2 \equiv b \cdots \bar{a}$ be alternating words in $\{a, b\}$ such that $S(z_2) = (4, 4, 5, 4)$ and $S(y_2) = (4, 5, 4, 4, 5, 4, 4, 5, 4, 4, 5, 4, 4, 5, 4, 4, 5, 4, 4)$, clearly $\bar{X}\bar{b}\bar{X}\bar{b}XbX \equiv z_2 y_2$. Here, since $CS(y_2) = CS(r_0)$ and so $y_2 =_{G_0} 1$, we finally have $\bar{X}\bar{b}\bar{X}\bar{b}XbX \equiv z_2 y_2 =_{G_0} z_2$, as required. \square

By Claims 1 and 2, it follows that

$$\begin{aligned} w_1 &=_{G_0} z_1^2 \equiv: w'_1, \\ w_2 &=_{G_0} z_2 z_1^{-1} \equiv: w'_2, \\ w_3 &=_{G_0} bX\bar{b}z_1 \equiv: w'_3, \\ w_4 &=_{G_0} z_2 z_1^{-1} b\bar{X}\bar{b} \equiv: w'_4, \\ w_5 &= w_2^{-1} =_{G_0} (w'_2)^{-1} \equiv: w'_5, \\ w_6 &= w_3^{-1} =_{G_0} (w'_3)^{-1} \equiv: w'_6, \end{aligned}$$

so that

$$(f(u_{r_0})) = (w_1 \cdots w_6) =_{G_0} (w'_1 \cdots w'_6).$$

Moreover, we see that $w'_1 \equiv a \cdots \bar{b}$, $w'_2 \equiv a \cdots \bar{a}$, $w'_3 \equiv b \cdots \bar{b}$, $w'_4 \equiv a \cdots \bar{b}$, $w'_5 \equiv a \cdots \bar{a}$ and $w'_6 \equiv b \cdots \bar{b}$ are alternating words in $\{a, b\}$ such that

$$\begin{aligned} S(w'_1) &= (S(z_1), S(z_1)) = (4, 5, 4, 5, 4, 5, 4, 5), \\ S(w'_2) &= (S(z_2), S(z_1^{-1})) = (4, 4, 5, 4, 5, 4, 5, 4), \\ S(w'_3) &= (S(bX\bar{b}), S(z_1)) = (5, 4, 4, 5, 4, 5, 4, 5), \\ S(w'_4) &= (S(z_2), S(z_1^{-1}), S(b\bar{X}\bar{b})) = (4, 4, 5, 4, 5, 4, 5, 4, 5, 4, 5, 4, 4, 5), \\ S(w'_5) &= S((w'_2)^{-1}) = (4, 5, 4, 5, 4, 5, 4, 4), \\ S(w'_6) &= S((w'_3)^{-1}) = (5, 4, 5, 4, 5, 4, 5, 4, 4, 5). \end{aligned}$$

This implies that

$$CS(w'_1 \cdots w'_6) = ((S(w'_1), \dots, S(w'_6))).$$

Following Notation 3.2, we also have

$$\begin{aligned} T(w'_1) &= (1, 1, 1, 1), & T(w'_2) &= (2, 1, 1, 1), \\ T(w'_3) &= (2, 1, 1, 1), & T(w'_4) &= (2, 1, 1, 1, 1, 2), \\ T(w'_5) &= (1, 1, 1, 2), & T(w'_6) &= (1, 1, 1, 2), \end{aligned}$$

and that

$$CT(w'_1 \cdots w'_6) = ((T(w'_1), \dots, T(w'_6))).$$

We furthermore have

$$\begin{aligned} V(w'_1) &= (4), & V(w'_2) &= (3), & V(w'_3) &= (3), \\ V(w'_4) &= (4), & V(w'_5) &= (3), & V(w'_6) &= (3), \end{aligned}$$

and

$$CV(w'_1 \cdots w'_6) = ((V(w'_1), \dots, V(w'_6))) = ((4, 3, 3, 4, 3, 3)).$$

Since $((4, 3, 3, 4, 3, 3))$ is the CS -sequence corresponding to the rational number $[3, 3]$, we see that

$$(w'_1 \cdots w'_6) \equiv (u_r^{\pm 1})$$

for some rational number r with $r'' = [3, 3]$. For this rational number r , since $CS(r') = CT(r) = CT(w'_1 \cdots w'_6)$ consists of 1 and 2, we have $r' = [1, 4, 3]$. Furthermore since $CS(r) = CS(w'_1 \cdots w'_6)$ consists of 4 and 5, we finally have $r = [4, 2, 4, 3]$ which equals r_1 in the statement of the theorem. This completes the proof of Lemma 3.3. \square

Lemma 3.4. *Under the foregoing notation, $(f(u_{r_i})) =_{G_0} (u_{r_{i+1}}^{\pm 1})$ for every $i \geq 1$.*

Proof. Fix $i \geq 1$. Then $r_i = [4, 2, m_3, \dots, m_k]$ with $m_3 \geq 3$. By Lemma 2.3(2), $CS(r_i)$ consists of 4 and 5 without (5, 5). Moreover, since $r'_i = [1, m_3, \dots, m_k]$, by Lemmas 2.3(2) and 2.5, $CT(r_i) = CS(r'_i)$ consists of 1 and 2, which implies that the number of occurrences of 4's between any two 5's is one or two.

Claim. By cutting the cyclic word (u_{r_i}) in the middle of each positive or negative subwords of length 5, we may write (u_{r_i}) as a product $(v_{i,1} \cdots v_{i,k_i})$, where each $v_{i,j}$ is one of the following:

$$\begin{aligned}
v_1 &\equiv \bar{b}abab\bar{a}\bar{b}\bar{a}\bar{b}ab, \\
v_2 &\equiv v_1^{-1} \equiv \bar{b}\bar{a}\bar{b}abab\bar{a}\bar{b}\bar{a}\bar{b}, \\
v_3 &\equiv abab\bar{a}\bar{b}\bar{a}\bar{b}abab\bar{a}\bar{b}\bar{a}, \\
v_4 &\equiv v_3^{-1} \equiv ab\bar{a}\bar{b}\bar{a}\bar{b}abab\bar{a}\bar{b}\bar{a}, \\
v_5 &\equiv \bar{b}abab\bar{a}\bar{b}\bar{a}, \\
v_6 &\equiv \bar{b}\bar{a}\bar{b}abab\bar{a}\bar{b}\bar{a}, \\
v_7 &\equiv v_6^{-1} \equiv abab\bar{a}\bar{b}\bar{a}\bar{b}ab, \\
v_8 &\equiv v_5^{-1} \equiv ab\bar{a}\bar{b}\bar{a}\bar{b}.
\end{aligned}$$

Proof of Claim. Note that for every $n = 1, \dots, 8$, v_n is an alternating word in $\{a, b\}$ such that $S(v_n) = (k_n, t_n \langle 4 \rangle, \ell_n)$, where $t_n \in \{1, 2\}$ and $k_n, \ell_n \in \{1, 2, 3, 4\}$. Consider the graph as in Figure 1, where the vertex set is equal to $\{v_1, \dots, v_8\}$ and each edge is endowed with one or two orientations. Observe that if v_n and v_m are the initial and terminal vertices, respectively, of an oriented edge of the graph, then the word $v_n v_m$ is an alternating word such that $S(v_n v_m) = (k_n, t_n \langle 4 \rangle, 5, t_m \langle 4 \rangle, \ell_m)$, namely, the terminal subword of v_n , corresponding to the last component ℓ_n of $S(v_n)$, and the initial subword of v_m , corresponding to the first component k_m of $S(v_m)$, are amalgamated into a maximal positive or negative alternating subword of $v_n v_m$, of length 5. Moreover, the weight t_n (resp. t_m) is 1 or 2 according to whether the vertex v_n (resp. v_m) has valence 3 or 2. Thus, if $v_{n_1}, v_{n_2}, \dots, v_{n_p}$, where $v_{n_j} \in \{v_1, \dots, v_8\}$, is a closed edge path in the graph which is compatible with the specified edge orientations (a compatible closed edge path, in brief), namely, if v_{n_j} and $v_{n_{j+1}}$ are the initial and terminal vertices of an oriented edge of the graph for each $j = 1, 2, \dots, p$, where the indices are considered modulo p , then the cyclically reduced word $v_{n_1} v_{n_2} \cdots v_{n_p}$ is a cyclically alternating word with CS -sequence $((5, t_{n_1} \langle 4 \rangle, 5, t_{n_2} \langle 4 \rangle, 5, \dots, t_{n_p} \langle 4 \rangle))$.

Since the weight t_{n_j} is 1 or 2 according to whether the vertex v_{n_j} has valence 3 or 2, we see that for any compatible closed edge path, the CT -sequence $((t_{n_1}, t_{n_1}, \dots, t_{n_p}))$ of the corresponding cyclically alternating word consists of only 1 and 2 and that it has isolated 2's. Moreover, for any such cyclic sequence, we can construct a compatible closed edge path such that the CT -sequence of the corresponding cyclically alternating word is equal to the given cyclic sequence. In particular, we can find a compatible closed edge path such that the CT -sequence of the corresponding cyclically alternating word, (w) , is equal to $CT(u_{r_i})$. This implies that $CS(w) = CS(u_{r_i})$.

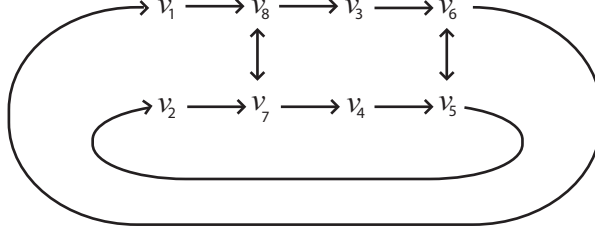


FIGURE 1. Proof of Claim in the proof of Lemma 3.4

Hence $(w) \equiv (u_{r_i}^{\pm 1})$ as cyclic words by [9, Lemma 3.2]. This completes the proof of Claim. \square

Putting

$$w_1 \equiv b\bar{X}\bar{b}\bar{X}\bar{b}XbXb\bar{X}\bar{b},$$

$$w_3 \equiv \bar{X}\bar{b}\bar{X}\bar{b}XbXb\bar{X}\bar{b}\bar{X}\bar{b}XbX,$$

$$w_5 \equiv b\bar{X}\bar{b}\bar{X}\bar{b}XbX,$$

$$w_6 \equiv bXb\bar{X}\bar{b}\bar{X}bXbX,$$

$$w_2 \equiv w_1^{-1}, \quad w_4 \equiv w_2^{-1}, \quad w_7 \equiv w_6^{-1} \quad \text{and} \quad w_8 \equiv w_5^{-1},$$

we obviously have $f(v_n) = w_n$ for every $n = 1, 2, \dots, 8$, so that

$$(f(u_{r_i})) = (f(v_{i,1} \cdots v_{i,k_i})) = (w_{i,1} \cdots w_{i,k_i}),$$

where each $w_{i,j} \in \{w_1, \dots, w_8\}$.

Recall from Claims 1 and 2 in the proof of Lemma 3.3 that $\bar{X}\bar{b}\bar{X}\bar{b}XbX\bar{b} =_{G_0} z_1$, where $z_1 \equiv a \cdots \bar{b}$ is the alternating word in $\{a, b\}$ with $S(z_1) = (4, 5, 4, 5)$, and that $\bar{X}\bar{b}\bar{X}\bar{b}XbX =_{G_0} z_2$, where $z_2 \equiv a \cdots \bar{a}$ is the alternating word in $\{a, b\}$ with $S(z_2) = (4, 4, 5, 4)$. It follows that

$$w_1 =_{G_0} z_1^{-1} b \bar{X} \bar{b} \equiv: w'_1,$$

$$w_3 =_{G_0} z_2 z_1^{-1} \equiv: w'_3,$$

$$w_5 =_{G_0} z_1^{-1} \equiv: w'_5,$$

$$w_6 =_{G_0} b X \bar{b} z_2^{-1} \equiv: w'_6,$$

$$w_2 = w_1^{-1} =_{G_0} (w'_1)^{-1} \equiv: w'_2,$$

$$w_4 = w_3^{-1} =_{G_0} (w'_3)^{-1} \equiv: w'_4,$$

$$w_7 = w_6^{-1} =_{G_0} (w'_6)^{-1} \equiv: w'_7,$$

$$w_8 = w_5^{-1} =_{G_0} (w'_5)^{-1} \equiv: w'_8.$$

Then we have

$$(f(u_{r_i})) =_{G_0} (w'_{i,1} \cdots w'_{i,k_i}),$$

where each $w'_{i,j} \in \{w'_1, \dots, w'_8\}$. Moreover

$$\begin{aligned} w'_1 &\equiv b \cdots \bar{b}, & w'_2 &\equiv b \cdots \bar{b}, & w'_3 &\equiv a \cdots \bar{a}, & w'_4 &\equiv a \cdots \bar{a}, \\ w'_5 &\equiv b \cdots \bar{a}, & w'_6 &\equiv b \cdots \bar{a}, & w'_7 &\equiv a \cdots \bar{b}, & w'_8 &\equiv a \cdots \bar{b} \end{aligned}$$

are alternating words in $\{a, b\}$ such that

$$\begin{aligned} S(w'_1) &= (S(z_1^{-1}), S(b\bar{X}\bar{b})) = (5, 4, 5, 4, 5, 4, 5, 4, 4, 5), \\ S(w'_3) &= (S(z_2), S(z_1^{-1})) = (4, 4, 5, 4, 5, 4, 5, 4), \\ S(w'_5) &= S(z_1^{-1}) = (5, 4, 5, 4), \\ S(w'_6) &= (S(bX\bar{b}), S(z_2^{-1})) = (5, 4, 4, 5, 4, 5, 4, 5, 4, 4), \end{aligned}$$

and $S(w'_2) = S((w'_1)^{-1}) = (5, 4, 4, 5, 4, 5, 4, 5, 4, 5)$, $S(w'_4) = S((w'_3)^{-1}) = (4, 5, 4, 5, 4, 5, 4, 4)$, $S(w'_7) = S((w'_6)^{-1}) = (4, 4, 5, 4, 5, 4, 5, 4, 4, 5)$, and $S(w'_8) = S((w'_5)^{-1}) = (4, 5, 4, 5)$.

Observe in the graph in Figure 1 that if v_n and v_m are the initial and terminal vertices, respectively, of an oriented edge, then $w'_n w'_m$ is an alternating word such that $S(w'_n w'_m) = (S(w'_n), S(w'_m))$, which consists of 4 and 5, and moreover the components 5 are isolated. This observation yields that

$$\begin{aligned} CS(w'_{i,1} \cdots w'_{i,k_i}) &= ((S(w'_{i,1}), \dots, S(w'_{i,k_i}))), \\ CT(w'_{i,1} \cdots w'_{i,k_i}) &= ((T(w'_{i,1}), \dots, T(w'_{i,k_i}))). \end{aligned}$$

Here

$$\begin{aligned} T(w'_1) &= (1, 1, 1, 2), & T(w'_2) &= (2, 1, 1, 1), \\ T(w'_3) &= (2, 1, 1, 1), & T(w'_4) &= (1, 1, 1, 2), \\ T(w'_5) &= (1, 1), & T(w'_6) &= (2, 1, 1, 2), \\ T(w'_7) &= (2, 1, 1, 2), & T(w'_8) &= (1, 1). \end{aligned}$$

This also yields that

$$CV(w'_{i,1} \cdots w'_{i,k_i}) = ((V(w'_{i,1}), \dots, V(w'_{i,k_i}))),$$

where $V(w'_n) = (3)$ if $n = 1, 2, 3, 4$, and $V(w'_n) = (2)$ otherwise.

Define $N(v_n)$ to be the number of positive or negative proper subwords of v_n of length 4 for each $n = 1, \dots, 8$. Here, by a proper subword of v_n , we mean a subword which lies in the interior of v_n . Then we see that $V(w'_n) = (N(v_n) + 1)$ for each $n = 1, \dots, 8$. Since $(v_{i,1} \cdots v_{i,k_i})$ is a product being cut in the middle of each positive or negative subwords of length 5, we also see that

$$((N(v_{i,1}), \dots, N(v_{i,k_i}))) = CT(r_i) = CS(r'_i)$$

with $r'_i = [1, m_3, \dots, m_k]$. Since $V(w'_{i,j}) = (N(v_{i,j}) + 1)$ for each $j = 1, \dots, k_i$, $CV(w'_{i,1} \cdots w'_{i,k_i}) = ((N(v_{i,1}) + 1, \dots, N(v_{i,k_i}) + 1))$ is the CS -sequence corresponding to the rational number $[2, m_3, \dots, m_k]$. Hence

$$(f(u_{r_i})) =_{G_0} (w'_{i,1} \cdots w'_{i,k_i}) \equiv (u_r^{\pm 1})$$

for some rational number r with $r'' = [2, m_3, \dots, m_k]$. For this rational number r , since $CS(r') = CT(r) = CT(w'_{i,1} \cdots w'_{i,k_i})$ consists of 1 and 2, we have $r' = [1, 3, m_3, \dots, m_k]$. Furthermore, since $CS(r) = CS(w'_{i,1} \cdots w'_{i,k_i})$ consists of 4 and 5, we finally have $r = [4, 2, 3, m_3, \dots, m_k]$ which equals r_{i+1} in the statement of the theorem. This completes the proof of Lemma 3.4. \square

Since $G = \langle a, b \mid u_{r_0} = u_{r_1} = u_{r_2} = \cdots = 1 \rangle$, Lemmas 3.1–3.4 imply that f descends to an epimorphism $\hat{f}: G \rightarrow G$. Now we show that \hat{f} is not an isomorphism. Let $s = [3, 3, 4]$. Then

$$CS(u_s) = CS(s) = ((3, 3, 3, 4, 3, 3, 4, 3, 3, 4, 3, 3, 4, 3, 3, 3, 4, 3, 3, 4, 3, 3, 4, 3, 3, 4)),$$

so that

[illegible]

As in the proof of Lemma 3.1, letting $w \equiv a \cdots \bar{a}$ be an alternating word in $\{a, b\}$ such that $S(w) = (3, 3, 3, 4, 3, 3)$, we have

$$(u_s) \equiv (wbwbab\bar{a}\bar{b}w^{-1}\bar{b}w^{-1}\bar{b}).$$

Lemma 3.5. *We have $\hat{f}(u_s) = 1$.*

Proof. Clearly

$$(f(u_s)) = (f(w)\bar{b}f(w)\bar{b}\bar{X}\bar{b}\bar{X}bXbf(w^{-1})bf(w^{-1})b).$$

Here, since $\hat{f}(w) = \bar{a}$ from the proof of Lemma 3.1, we have

$$(f(u_s)) =_G (\bar{a}\bar{b}\bar{a}\bar{b}\bar{X}\bar{b}\bar{X}bXbabab),$$

where $(\bar{a}\bar{b}\bar{a}\bar{b}\bar{X}\bar{b}\bar{X}\bar{b}Xbabab)$ is a cyclically alternating word in $\{a, b\}$ such that

$$CS(\bar{a}\bar{b}\bar{a}\bar{b}\bar{X}\bar{b}\bar{X}\bar{b}Xbabab) = ((S(\bar{a}\bar{b}\bar{a}\bar{b}), S(\bar{X}\bar{b}), S(\bar{X}), S(bX), S(babab))), \\ = ((4, (4, 4, 5, 4, 4, 5), (4, 4, 5, 4, 4, 4), (5, 4, 4, 5, 4, 4), 5))$$

which equals $CS(r_0)$. This implies that $(f(u_s)) =_G (\bar{a}\bar{b}\bar{a}\bar{b}\bar{X}\bar{b}\bar{X}bXbabab) =_G 1$, namely $\hat{f}(u_s) = 1$, as required. \square

Lemma 3.6. *Under the foregoing notation, let R be the symmetrized subset of $F(a, b)$ generated by the set of relators $\{u_{r_i} \mid i \geq 0\}$ of the upper presentation $G = \langle a, b \mid u_{r_0} = u_{r_1} = u_{r_2} = \cdots = 1 \rangle$. Then R satisfies $C(4)$ and $T(4)$.*

Proof. Since every element in R is cyclically alternating, R clearly satisfies $T(4)$. To show that R satisfies $C(4)$, we begin by setting some notation. Recall from Lemma 2.6 that for every rational number r with $0 < r \leq 1$, $CS(r)$ has a decomposition $((S_1, S_2, S_1, S_2))$ depending on r . For clarity, we write $((S_1(r), S_2(r), S_1(r), S_2(r)))$ for this decomposition. On the other hand, if r is a rational number with $r = [m_1, \dots, m_k]$ with $k \geq 2$ and $(m_2, \dots, m_k) \in (\mathbb{Z}_{\geq 2})^{k-1}$, then the symbol $r^{(n)}$ denotes the rational number with continued fraction expansion $[m_{n+1} - 1, m_{n+2}, \dots, m_k]$ for each $n = 1, \dots, k - 1$.

Claim 1. *For any two integers $i, j \geq 0$ with $i \neq j$, the cyclic word (u_{r_j}) does not contain a subword corresponding to $(S_1(r_i))$ or $(\ell_1, S_2(r_i), \ell_2)$ with $\ell_1, \ell_2 \in \mathbb{Z}_+$.*

Proof of Claim 1. Suppose on the contrary that there are some $i \neq j$ such that the cyclic word (u_{r_j}) contains a subword corresponding to $S_1(r_i)$ or $(\ell_1, S_2(r_i), \ell_2)$. We first show that this assumption implies that $CS(r_j)$ contains $S_1(r_i)$ or $S_2(r_i)$ as a subsequence. If (u_{r_j}) contains a subword corresponding to $(\ell_1, S_2(r_i), \ell_2)$, then clearly $CS(r_j)$ contains $S_2(r_i)$ as a subsequence. So assume that (u_{r_j}) contains a subword corresponding to $S_1(r_i)$. Then $CS(r_j)$ contains $(d_1 + s_1, s_2, \dots, s_{t-1}, s_t + d_2)$ as a subsequence, where $S_1(r_i) = (s_1, s_2, \dots, s_t)$. Since the continued fraction expansions of both r_i and r_j begin with 4, we see that $S_1(r_i)$ begins and ends with 5 by Lemma 2.6(3) and that $CS(r_j)$ also consists of 4 and 5 by Lemma 2.3(2). Hence, we must have $d_1 = d_2 = 0$ and therefore $CS(r_j)$ contains $S_1(r_i)$ as a subsequence. Thus we have proved that $CS(r_j)$ contains $S_1(r_i)$ or $S_2(r_i)$ as a subsequence.

Note that the lengths of the continued fraction expansions of r_i and r_j are $i+3$ and $j+3$, respectively. Hence we can apply Lemma 2.8 successively to see that $CS(r_j^{(n)})$ contains $S_1(r_i^{(n)})$ or $S_2(r_i^{(n)})$ as a subsequence for every $n = 1, \dots, \min\{i+1, j+1\}$. Since $i \neq j$, there are two cases.

Case 1. $j > i \geq 0$. Recall that r_i is equal to $[4, 3, 3]$ or $r_i = [4, 2, (i-1)\langle 3 \rangle, 4, 3]$ according to whether $i = 0$ or $i \geq 1$. So we have $r_i^{(i+1)} = [m, 3]$. Here, $m = 2$ if $i = 0$, and $m = 3$ otherwise. Since $j > i$, we can observe that $r_j^{(i+1)}$ has a continued fraction expansion of the form $[m-1, n_1, \dots, n_k]$, where $k \geq 2$ and each n_t is 3 or 4. Since $S_1(r_i^{(i+1)}) = (m+1)$ and $CS(r_j^{(i+1)})$ consists of $m-1$ and m , the cyclic sequence $CS(r_j^{(i+1)})$ cannot contain $S_1(r_i^{(i+1)}) = (m+1)$ as a subsequence. Hence $CS(r_j^{(i+1)})$ must contain $S_2(r_i^{(i+1)})$ as a subsequence. But since $r_j^{(i+1)} = [m-1, n_1, \dots, n_k]$ with $n_1 \geq 3$, (m, m) does not occur in $CS(r_j^{(i+1)})$ by Lemma 2.3(2). Since $S_2(r_i^{(i+1)}) = (m, m)$ by Lemma 2.7(1), this implies that $S_2(r_i^{(i+1)})$ cannot occur in $CS(r_j^{(i+1)})$, a contradiction.

Case 2. $i > j \geq 0$. As in Case 1, we can observe that $r_j^{(j+1)} = [m, 3]$, where $m = 2$ if $j = 0$, and $m = 3$ otherwise, and that $r_i^{(j+1)}$ has a continued fraction expansion of

the form $[m-1, n_1, \dots, n_k]$, where $k \geq 2$ and each n_t is 3 or 4. Then both $S_1(r_i^{(j+1)})$ and $S_2(r_i^{(j+1)})$ contain a term $m-1$ by Lemma 2.7(2). But since $CS(r_j^{(j+1)})$ consists of only m and $m+1$, this is impossible. \square

By Claim 1, we see that the assertion in Lemma 2.11 holds even if $(u_r^{\pm 1})$ is replaced by $(u_{r_i}^{\pm 1})$ for any $i \geq 0$ and the symmetrized subset R in the lemma is enlarged to be the set in the current setting, namely, R is the symmetrized subset of $F(a, b)$ generated by the set of relators $\{u_{r_i} \mid i \geq 0\}$ of the group presentation $G = \langle a, b \mid u_{r_0} = u_{r_1} = u_{r_2} = \dots = 1 \rangle$. To be precise, the following hold.

Claim 2. *For each $i \geq 0$, a subword w of the cyclic word $(u_{r_i}^{\pm 1})$ is a piece with respect to the symmetrized subset R in Lemma 3.6 if and only if $S(w)$ contains neither $S_1(r_i)$ nor $(\ell_1, S_2(r_i), \ell_2)$ with $\ell_1, \ell_2 \in \mathbb{Z}_+$ as a subsequence.*

By using Claim 2, we can see, as in [9, Proof of Corollary 5.4], that each cyclic word $(u_{r_i}^{\pm 1})$ is not a product of less than 4 pieces with respect to R . Hence R satisfies $C(4)$. \square

Lemma 3.7. *Under the foregoing notation, $u_s \neq_G 1$.*

Proof. Suppose on the contrary that $u_s =_G 1$. Then there is a reduced van Kampen diagram Δ over $G = \langle a, b \mid u_{r_0} = u_{r_1} = u_{r_2} = \dots = 1 \rangle$ such that $(\phi(\partial\Delta)) \equiv (u_s)$ (see [11]). Since Δ is a $[4, 4]$ -map by Lemma 3.6, $(\phi(\partial\Delta))$ contains a subword of some $(u_{r_i}^{\pm 1})$ which is a product of 3 pieces with respect to the symmetrized subset R in Lemma 3.6 (see [9, Section 6]). This implies that $CS(\phi(\partial\Delta))$ must contain a term 5, which is a contradiction to the fact $CS(\phi(\partial\Delta)) = CS(u_s) = CS(s)$ consists of only 3 and 4. \square

Lemma 3.7 together with Lemma 3.5 shows that \hat{f} is an epimorphism of G , but not an isomorphism of G . Consequently, G is non-Hopfian, and the proof of Theorem 1.1 is now completed. \square

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